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APPLICATIONS OF THE MILLER SPECTRAL SEQUENCE

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1. INTRODUCTION. A k-connected Ω spectrum $\underline{X}$ is a sequence of n + k connected compactly generated Hausdorff spaces X_n with nondegenerate basepoints, together with based homotopy equivalences $X_n \simeq \Omega X_{n+1}$. See [A2] and [M1] for more details. For example, if Y is a k-connected space, then $\underline{\Sigma^\infty Y}$ is the k-connected Ω spectrum with n^{th} space $Q\Sigma^n Y = \lim_N \Omega^N \Sigma^{N+n} Y$. If $\underline{X}$ is a k-connected Ω spectrum, then $\underline{\Sigma X}$ is the k + 1 connected Ω spectrum with n^{th} space X_{n+1}. If π is an abelian group, then $\underline{K(\pi)}$ is the -1 connected Ω spectrum with n^{th} space $K(\pi,n)$.

There are -1 connected Ω spectra $\underline{ko}$ and $\underline{ku}$ with 0^{th} spaces Z x BO and Z x BU which have proved quite important in many aspects of algebraic topology [A2]. In this paper we will be especially interested in the 0 connected spectrum $\underline{\Sigma ko}$ with 0^{th} space B(Z x BO) = U/O.

Let G be the space of stable self equivalences of spheres. There is a natural map $J: O \longrightarrow G$ where O is the infinite orthogonal group given by restricting orthogonal transformations on R^{n+1} to S^n. Boardman and Vogt [BV] have shown that G is the 0^{th} space of an Ω spectrum g and that J induces a map of Ω spectra. Thus there is a 1 connected Ω spectrum g/o whose 0^{th} space is G/O. The study of the spectrum g/o should have implications in stable homotopy theory. For example, $\pi_k(g/o) = \pi_k(G/O)$ is closely related to the stable stem.

Throughout this paper all homology and cohomology groups will have Z/2 coefficients. If $\underline{X}$ is a -1 connected Ω spectrum, then define $H_q(\underline{X}) = \lim_n H_{q+n}(X_n)$. Connectivity implies that $H_q(\underline{X}) = H_{q+n}(X_n)$ for n > q. Similarly define $H^q(\underline{X}) = H^{q+n}(X_n)$ for n > q.

Examples:

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(1.1) $\tilde{H}_q(\underline{\Sigma^\infty Y}) = \tilde{H}_{q+n}(Q\Sigma^n Y)$
$\approx \tilde{H}_{q+n}(\Sigma^n Y)$
$\approx \tilde{H}_q(Y)$.

(1.2) $H^*(K(Z/2)) \approx \mathcal{a}$, the mod 2 Steenrod algebra.

(1.3) $H^*(\underline{ku}) \approx \mathcal{a}/\mathcal{a}(Sq^1, Sq^3)$ [A1].

(1.4) $H^*(\underline{ko}) \approx \mathcal{a}/\mathcal{a}(Sq^1, Sq^2)$ [S].

(1.5) $H^q(\Sigma^k X) = S^k H^q(X) = H^{q-k}(X)$.

Haynes Miller and Stewart Priddy [MP] conjectured the existence of a fibration of infinite loop spaces relating G/O to a stable fibration over QS^o. A corollary of their conjecture is that

(1.6) $H^*(g/o) \approx S^2\ \mathcal{a}/\mathcal{a}(Sq^1, Sq^3) \oplus S^{-1}\tilde{\mathcal{a}}/\mathcal{a}(Sq^1, Sq^2)$

as Z/2 modules where $\tilde{\mathcal{a}}$ is the augmentation ideal of $\mathcal{a}$. In the next section we will describe the Miller spectral sequence or (MSS) which converges to the cohomology of 0-connected spectra. The authors have been able to show that the E_2 term of the MSS converging to $H^*(g/o)$ is isomorphic to the module in (1.6). In this paper we give a crucial step in this proof, namely the computation of the MSS converging to $H^*(\Sigma\underline{ko})$.

2. THE MILLER SPECTRAL SEQUENCE. The Miller delooping spectral sequence converges to the homology or cohomology of -1 connected Ω spectra. If $\underline{X}$ is a 0-connected spectrum with 0^{th} space X such that $H_*(X)$ is a polynomial algebra, then Miller has described the homology E^2 term as an unstable Tor functor over the Dyer Lashof algebra $\mathcal{R}$ on the quotient module $QH_*(X)$ of indecomposables. In this section we outline his method for computing E^2. More precisely, we describe a bigraded complex $L^{p,q}(X)$ whose cohomology is the cohomology $E_2^{p,q}(X)$ term of the MSS converging to $H^*(\underline{X})$.

<u>Definition 2.1</u>. Let L be the associative algebra on symbols $\sigma(a)$ for $a \geq 1$, where we abbreviate $\sigma(a_1,\ldots,a_p) = \sigma(a_1)\ldots\sigma(a_p)$, subject to the (Adem) relations

$$\sigma(a,b) = \sum_j \binom{b-1-j}{a-2j} \sigma(a+b-j,j).$$

Let L(n) be the submodule of L generated by sequences $\sigma(I)$ for $I = (i_1+1,\ldots,i_p+1)$ satisfying $i_j > i_{j+1}+\ldots+i_p + n$ for $1 \leq j \leq p$. Thus L(n) is isomorphic to a submodule of the Steenrod algebra via the assignment $\sigma(a_1,\ldots,a_p) \mapsto Sq^{a_1}\ldots Sq^{a_p}$. This implies that a basis for $L(n)$ consists of admissible sequences $\sigma(a_1,\ldots,a_s)$ with $a_s > n + 1$, i.e. sequences with $a_j \geq 2a_{j+1}$. Note that $L(n)$ is closed under the Adem relations. For example, if $2b > a \geq b + n + 1$, then it is easy to check that $\binom{b-1-j}{a-2j} = 0$ if $j < a - b \leq n + 1$. The relations (3.3.7) in [M3] are known to be equivalent to these. See the proof of Lemma 4.5 below.

If X is an infinite loop space then $H^*(X)$ has an adjoint Dyer Lashof action $Q^r_*: H^n(X) \longrightarrow H^{n-r}(X)$. By the Cartan formula for homology operations this action clearly extends to $PH^*(X)$, the module of primitive cohomology classes of X. For more details see [M2].

Definition 2.2. Let X be a connected infinite loop space. Then $L^{p,q}(X)$ is the bimodule $\bigoplus_{n \geq 0} L(n) \otimes PH^n(X)$ where bideg $\sigma(a_1,\ldots,a_p)\mu = (p, \Sigma a_j + n)$ if $\mu \in PH^n(X)$. The differential is given by

$$d\sigma(I)\mu = \sum_{r=[\frac{n}{2}]}^{n-1} \sigma(I, r + 1)Q^r_* \mu.$$

Note that since $Q^r x = 0$ if $r < \dim x$, we have $Q^r_* \mu = 0$ if $r < n - r$. Thus $\sigma(I, r + 1) \in L(n - r)$ and thus d is well defined. See [M3] for more details.

Theorem 2.3. (Miller). If X is a 0-connected infinite loop space and if $H_*(X)$ is a polynomial algebra over $Z/2$, then $E_2^{p,q}(X) \approx H^{p,q}(L(X))$.

Proof. See Theorem 3.3.16 and p. 144 [M3].

Remark 2.4. In a forthcoming paper [KL2] we generalize this theorem to arbitrary connected infinite loop spaces with coefficients in Z/p. Furthermore we apply this to get results on the Miller spectral sequence of Postnikov systems and to construct and evaluate higher order Dyer Lashof operations.

If $X = QY$ or if $X = K(Z/2, n)$, then the MSS is fairly easy to compute. Except in these situations and minor modifications of them, only partial computations have appeared (see [M3], [KL1]). The main result of this paper is the computation of $E_2(U/O)$. This result can be applied directly to describe the MSS for other spaces such as BU, SU, etc. Most importantly, using a theorem of Tornehave [T], we can show that $E_2(G/O)$ can be easily computed via these results. Indeed $E_2(G/O)$ is isomorphic to the module in equation (1.6). Moreover there are very stringent algebraic restrictions on the higher differentials. The existence of nontrivial higher differentials appears to be related to the Arf invariant question. Details will appear later.

3. THE MILLER SPECTRAL SEQUENCE FOR U/O. First recall that $H_*(U/O) \approx Z/2\,[k_1,k_2,\ldots]$ where $\dim k_t = 2t-1$. Moreover the Dyer Lashof action is given by $Q^{2r}k_s = \binom{r-1}{s-1} k_{s-r}$ and $Q^{2r+1}k_s = 0$. [K1].

Thus $H^{2t-1}(U/O)$ has primitive classes κ_t with $\langle \kappa_t, k_s\rangle = \delta_{s,t}$. Moreover $\langle Q_*^{2r}\kappa_t, k_{t-r}\rangle = \langle \kappa_t, Q^{2r}k_{t-r}\rangle = \binom{r-1}{t-r-1}$. Thus $Q_*^{2r}\kappa_t = \binom{r-1}{t-r-1}\kappa_{t-r}$. We have thus proved the following description of the fundamental complex $L(U/O)$ for $E_2(U/O)$.

<u>Proposition 3.1.</u> A basis for $L^{p,q}(U/O)$ consists of admissible sequences $\sigma(a_1,\ldots,a_p)\kappa_t$. That is sequences satisfying $a_p \geq 2t+1$, $a_i \geq 2a_{i+1}$ for $i = 1,\ldots,p-1$, and $q = \Sigma a_j + 2t - 1$. The differential is given by

$$d\sigma(a_1,\ldots,a_p)\kappa_t = \sum_{r=[\frac{t}{2}]}^{t-1} \binom{r-1}{t-r-1} (a_1,\ldots,a_p,\ 2r+1)\kappa_{t-r}.$$

Note that $\sigma(I)\kappa_1$ is a cocycle for all I. Also the summand with $t - r = 1$ is always nonzero if $t > 1$. The expression $\sigma(a_1,\ldots,a_p,\ 2r+1)$ is not in general admissible. For example $d\sigma(5)\kappa_2 = \sigma(5,3)\kappa_1 = 0$.

Our main result is that $H(L(U/O)) = E_2(U/O)$ is isomorphic (at least as Z/2 modules) to $E_\infty \approx H^*(\underline{\Sigma ko}) \approx S\mathcal{a}/\mathcal{a}(Sq^1,Sq^2)$. In order to prove this, we must examine the ideal $\mathcal{a}(Sq^1,Sq^2)$, or more precisely a filtered version of it, more closely.

<u>Definition 3.2.</u> Let $G^k\mathcal{a} = \{Sq^I : I \text{ is admissible and } \ell(I) \geq k\}$. If M is a submodule of $\mathcal{a}$, then set $G^kM = M \cap G^k\mathcal{a}$. Finally let $G_oM = \bigoplus_{k\geq o} G^kM/G^{k+1}M$ be the associated graded object.

<u>Remark 3.3.</u> Note that there are isomorphisms of bigraded modules $L(-1) \approx G_o\mathcal{a}$ and $L(0) \approx (G_o\mathcal{a})/(G_o\mathcal{a}Sq^1)$.

<u>Proposition 3.4.</u> $G_o\mathcal{a}(Sq^1,Sq^2)$ is generated by elements of the following types:

a) $Sq^{a_1}\ldots Sq^{a_p}Sq^1$ with $(a_1,\ldots,a_p,1)$ admissible

b) $Sq^{a_1}\ldots Sq^{a_p}Sq^{2^k+1}$ with $(a_1,\ldots,a_p)$ admissible and $a_p > 2^k+1$ for $k \geq 0$.

<u>Remark 3.5.</u> We emphatically do not assert that these elements form a basis

for $\mathcal{a}(Sq^1,Sq^2)$. Indeed $Sq^9Sq^4Sq^3 = Sq^9Sq^5Sq^2 = 0$.

Proof. $\mathcal{a}(Sq^1,Sq^2)$ has generators Sq^ISq^1 and Sq^ISq^2 as $I = (a_1,\dots,a_p)$ ranges over admissible sequences. If $a_p = 1$, then $Sq^ISq^1 = 0$ while if $a_p > 1$, Sq^ISq^1 is admissible of type (a). If $a_p \geq 4$, then Sq^ISq^2 is admissible of type (b). If $a_p = 3$, $Sq^ISq^2 = 0$ and if $a_p = 2$, it is in $\mathcal{a}Sq^1$ since $Sq^2Sq^2 = Sq^3Sq^1$.

Assume finally that we can write $I = (a_1,\dots,a_{p-k},2^{k-1},\dots,2,1)$ with $a_{p-k} \geq 2^k + 1$. By the Adem relations, $Sq^{2^\ell}Sq^{2^\ell+1} = Sq^{2^{\ell+1}+1}$ + terms of length 2. (See also Lemma 4.1). Thus modulo terms of higher length, $Sq^ISq^2 = Sq^{a_1}\dots Sq^{a_{p-k}}Sq^{2^k+1}$.

By applying the derivation κ^{2^k} of Kristensen [K2] to the relation $Sq^{2^{k+1}+1}Sq^{2^k+1} = 0$, we see that $Sq^{2^k+1}Sq^{2^k+1} = Sq^{2^{k+1}+1}Sq^1$. This is equivalent to applying $\xi_1^{2^k}\cap$ where ξ_1 is the Milnor dual of Sq^1 and $\cap$ is the cap pairing $\mathcal{a}^m \otimes \mathcal{a}_n \longrightarrow \mathcal{a}_{n-m}$. See also the proof of Lemma 4.5. Thus if $a_{p-k} = 2^k + 1$, $Sq^ISq^2 \in \mathcal{a}Sq^1$ and so we may assume that $a_{p-k} > 2^k + 1$.

To compute the cohomology of $L(U/O)$, first define a filtration of it by letting $F_t = F_t\, L(U/O)$ be generated by $\sigma(I)\kappa_s$ for $s \leq t$. Note that $d(F_t) \subset F_{[\frac{t}{2}]} \subset F_{t-1}$.

<u>Definition 3.6.</u> Let

$$c: L^{p,q}(U/O) \longrightarrow L^{p-1,q}(U/O)$$

be defined inductively as follows:

1) $c(\kappa_t) = 0$ for all t

2) $c(\sigma(a)\kappa_t) = \begin{cases} 0 & \text{if } a \neq 2^{i+1}+1 \\ \kappa_{t+2^i} & \text{if } a = 2^{i+1}+1 \end{cases}$

3) $c\sigma(a,J)\kappa_t = \sigma(a)\ c\sigma(J)\kappa_t$

if $a \neq 2^{m+1} + 1$ and $\sigma(a,J)$ is admissible

4) If $\sigma(2^{m+1} + 1, b, K)$ is admissible then

$$c\sigma(2^{m+1} + 1, b, K)\kappa_t = \sigma(2^m + b)\ c\sigma(2^m + 1, K)\kappa_t$$

$$+ \sum_{j>b+1} \binom{2^m - j}{2^m + b - 2j} \sigma(2^{m+1} + 1 + b - j)\ c\sigma(j, K)\kappa_t$$

$$= \sum_{j=b+1}^{2^m+1} \gamma_j \sigma(2^{m+1} + 1 + b - j)\ c\sigma(j, K)\kappa_t$$

where $\gamma_j = \binom{2^m - j}{2^m + b - 2j}$ if $b + 1 \leq j \leq 2^m$ and $\gamma_{2^m+1} = 1$.

Remark 3.7. Some discussion of this definition is in order. First note that

$$\sigma(2^{m+1} + 1, b) = \sigma(2^m + b)\sigma(2^m + 1)$$

$$+ \sum_{j>a} \binom{2^m - j}{2^m + b - 2j} \sigma(2^{m+1} + 1 + b - j)\sigma(j)$$

is an Adem relation (on the second summand). Moreover this coefficient is 1 whenever $j = 2^s + 1$ in the dimension range by Lemma 4.1 below. Thus we may write

$$c\sigma(2^{m+1} + 1, b, K)\kappa_t = \sum \gamma_j \sigma(2^{m+1} + 1 + b - j)c(\sigma(j, K))\kappa_t$$

where $\gamma_j = 1$ if $j = 2^s + 1$ for $2^m + 1 \geq j > b$.

We will see that $d\kappa_t = \sigma(2^{i+1} + 1)\kappa_{t-2^i}$ + terms in which r is not a power of 2. Thus $cd\kappa_t = \kappa_t$. In general, to compute $c(\sigma(a,J))\kappa_t$ we can just push c over if $a \neq 2^m + 1$. To compute $c\sigma(2^{m+1} + 1, b, K)\kappa_t$, we just rewrite this expression as a sum of nonadmissible sequences none of which start with $2^s + 1$ and then push c over.

We now state the main technical theorem. The proof is quite long and will occupy sections 4 and 5. This theorem will be the key step in our future work of evaluating the MSS for related spaces such as BU and SU as well as the E_2 term for G/O.

Theorem 3.8. If $t > 1$, then

$$dc\sigma(I)\kappa_t + cd\sigma(I)\kappa_t = \sigma(I)\kappa_t \bmod F_{t-1}.$$

Corollary 3.9. Let $\alpha \varepsilon L(U/O)$ be a cocycle. Then α is cohomologous to a cocycle $\alpha' \varepsilon \Sigma\sigma(I_i)\kappa_1 \varepsilon F_1$.

Proof: Write $\alpha = \alpha_1 + \ldots + \alpha_t$ where $\alpha_j = \Sigma\sigma(I_{ij})\kappa_j$. If $t > 1$ then $dc\alpha + cd\alpha = dc\alpha = \alpha \bmod F_{t-1}$. This means that $d(c\alpha) = \alpha - \bar{\alpha}$ for some $\bar{\alpha} \varepsilon F_{t-1}$. The result follows by repeating this process $t - 1$ times.

We need only partial control over the boundaries in F_1 to evaluate $H(L(U/O)) = E_2(U/O)$.

Lemma 3.10. $d\sigma(I)\kappa_{1+2^s} = \sigma(I)\sigma(2^{s+1} + 1)\kappa_1$.

Proof. By definition of d it suffices to prove this for $I = \phi$. Note that

$$d\kappa_{1+2^s} = \sum \binom{r-1}{2^s-r} \sigma(2r+1)\kappa_{1+2^s-r} \quad \text{and that } \gamma = \binom{r-1}{2^s-r} = \binom{r-1}{2r-2^s-1}.$$

Clearly $\gamma = 1$ if $r = 2^s$. Also it is easily seen that $\gamma = 0$ if $r > 2^s$ or $r \leq 2^{s-1}$. Assume that $2^{s-1} < r < 2^s$ and write $r = 2^i(2b + 1)$ for $b \geq 2^{s-i-2} \geq 1$. Then $r - 1 = 1 + 2 + \ldots + 2^{i-1} + 2^{i+1}b$ while $2r - 1 - 2^s = 1 + 2 + \ldots + 2^i + 2^{i+2}(b - 2^{s-i-2})$. By comparison of the coefficients of 2^i we see that $\gamma = 0$ by Lemma 26 [SE].

We can finally state and prove the main result of this paper.

Theorem 3.11. There is an isomorphism $G_o\mathcal{a}/G_o(\mathcal{a}(Sq^1,Sq^2)) \longrightarrow E_2(U/O)$.

Proof. Let $\phi: G_o\mathcal{a} \to F_1L(U/O)$ be the degree 1 map given by $Sq^I \longrightarrow \sigma(I)\kappa_1$ where we set $\sigma(a_1,\ldots,a_p)\kappa_1 = 0$ if $a_p = 1$ or 2. Then ϕ is a surjection with kernel generated by admissible operations $Sq^{i_1}\ldots Sq^{i_p}$ with $i_p = 1$ or 2. By Proposition 3.4, $G_o\mathcal{a}(Sq^1,Sq^2)$ is generated by these elements together with elements $Sq^{a_1}\ldots Sq^{a_p}Sq^{2^{s+1}+1}$ with $a_p > 2^{s+1} + 1$. Since $\sigma(a_1,\ldots,a_p,2^{s+1}+1)\kappa_1 = d\sigma(a_1,\ldots,a_p)\kappa_{1+2^s}$, it follows that ϕ induces a surjection

$$\begin{aligned}\phi_o: G_o\mathcal{a}/G_o\mathcal{a}(Sq^1,Sq^2) &\longrightarrow F_1/\mathrm{Im}\, d \cap F_1 \\ &= H(L(U/O)).\end{aligned}$$

Thus $\sum_{p+q=n-1} \dim\, (G_o\mathcal{a}/G_o\mathcal{a}(Sq^1,Sq^2)^{p,q} \geq$

$$\sum_{p+q=n} \dim H^{p,q}(L(U/O)) = \sum_{p+q=n} \dim E_2^{p,q} \geq$$

$$\sum_{p+q=n} \dim E_\infty^{p,q} = \dim H^n(\Sigma \underline{ko}) = \dim (\mathcal{a}/\mathcal{a}(Sq^1,Sq^2)^{n-1}).$$

It follows that all the inequalities are equalities and thus ϕ_o is an isomorphism.

4. TECHNICAL LEMMAS. We start off with a result concerning the binomial coefficient appearing in the Adem relation for $Sq^{a+2^m} Sq^{2^m+1}$.

Lemma 4.1. Assume that $a > 1$. Then $\gamma_j = \binom{2^m-j}{a+2^m-2j} = 1 \pmod 2$ if $j = 2^s + 1$ and $2^m + 1 > 2^s + 1 > a$. If $a = 2^q + 1$ then $\gamma_j = 0 \pmod 2$ if $j \neq 2^s + 1$. Thus

$$\sum_{s=q}^{m} Sq^{2^{m+1} - 2^s + 2^q + 1} Sq^{2^s+1} = 0.$$

Proof. Using the equality $\binom{i}{j} = \binom{i}{i-j}$, first write

$\gamma_{2^s+1} = \binom{2^m-2^s-1}{2^s-a+1}$ and note that $2^m - 2^s - 1 = 1 + 2 + \ldots + 2^{s-1} + 2^{s+1}$ $+\ldots+ 2^{m-1}$. Since $a > 1$, we have that $2^s - a + 1 \leq 2^s - 1$ and so $\gamma_{2^s+1} = 1$ by Lemma 2.6 [SE].

Now assume that $a = 2^q + 1$ and that $j - 1$ is not a power of 2. Write $j - 1 = 2^x + 2^y + 2^{y+1}z$ for $x < y$ and $z \geq 0$. In the expansion $2^m - j = (2^m - 1) - (j - 1) = b_o +\ldots+ b_i 2^i +\ldots$ we see that $b_x = b_y = 0$. In the expansion $j - 1 - 2^q = c_o +\ldots+ c_i 2^i +\ldots$, it is easy to check that at least one of c_x or c_y is 1. Since $\binom{0}{1} = 0$, $\gamma_j = 0$ by the same lemma of [SE]. Finally the last equations follow from the above and the Adem relation for $Sq^{2^m+2^q+1} Sq^{2^m+1}$.

The next three results are used to compute the contraction c on a class of nonadmissible generators of L(U/O). Recall that one may identify L(n) with the underlying module of the Steenrod algebra provided that attention is restricted to sequences in L(n).

Lemma 4.2. Assume that $\sigma(a_1,\ldots,a_p) \in L$ is admissible and that $a_p > b + 1$

while $a_1 < 2^p b + 1$. Then $\sigma(a_1,\ldots,a_p, b+1) = \sigma(2^p b + 1, a_1 - 2^{p-1}b,\ldots, a_p - b) + \sum_\alpha \sigma(b_o^\alpha,\ldots,b_p^\alpha)$ in admissible form where the latter sum satisfies $b_o^\alpha < 2^p b + 1$ for all α.

Proof. Note that the inequalities $2^{p-1}a_p \le \ldots \le 2a_2 \le a_1 < 2^p b + 1$ imply that $a_p < 2b + 1$ so that $\sigma(a_p, b+1)$ is not admissible. If $p = 1$, then

$$\sigma(a, b+1) = \sum \binom{b-j}{a-2j} \sigma(a + b - j + 1, j).$$

The coefficient is 0 if $a - 2j > b - j$, that is if $a - b > j$, and it is 1 if $a - b = j$. Thus $\sigma(a, b+1) = \sigma(2b + 1, a - b)$ + terms of lower lexicographic order.

We now proceed by a double induction on b and p. If $b = 0$, then the lemma is vacuously true for all p. Assume that the lemma is true for all sequences of length $< p$ and for all sequences of length $= p$ with right hand entry $< b + 1$. Then $\sigma(a_1,\ldots,a_p, b+1) =$

$$\sigma(a_1,\ldots,a_{p-1}, 2b+1, a_p - b) + \sum_{b \ge j > a_p - b} \mu_j \sigma(a_1,\ldots,a_{p-1}, a_p + b - j, j)$$

for some numbers $\mu_j \in Z/2$.

By induction on p,

$$\sigma(a_1,\ldots,a_{p-1}, 2b+1) = \sigma(2^p b + 1, a_1 - 2^{p-1}b,\ldots,a_{p-1} - 2b) + \text{lower}$$

terms in lexicographic order. Also since $a_p + b - j < 2b + 1$, $\sigma(a_1,\ldots,a_p + b - j)$ is a sum of admissible terms with each leading entry less than $2^p b + 1$. Thus $\sigma(a_1,\ldots,a_p, b+1) = \sigma(2^p b + 1,\ldots,a_p - b)$

$$+ \sum_{\alpha,j} \sigma(b_o^\alpha,\ldots,b_{p-1}^\alpha, j)$$

where $\sigma(b_o^\alpha,\ldots,b_{p-1}^\alpha)$ is admissible, $b_o^\alpha < 2^p b + 1$ and $j < b + 1$. The lemma follows by induction on b.

Lemma 4.3. Assume that $\sigma(b_o,\ldots,b_p) \in L(2t - 1)$ is admissible. If $b_o < 2^{m+1} + 1$, then $c\sigma(b_o,\ldots,b_p)\kappa_t \in F_{t+2^{m-p-1}} \subset F_{t+2^{m-p}-1}$. If $b_o = 2^{m+1} + 1$, then

$$c\sigma(2^{m+1} + 1, b_1,\ldots,b_p)\kappa_t = \sigma(b_1 + 2^m,\ldots,b_p + 2^{m-p+1})\kappa_{t+2^{m-p}}$$

modulo $F_{t+2^{m-p}-1}$.

Proof. If $p = 0$ then each statement follows immediately from Definition 3.6. Assume that the lemma is true for sequences of length $< p$. If $2^m + 1 < b_o < 2^{m+1} + 1$, then again by Definition 3.6

$$c\sigma(b_o,\ldots,b_p)\kappa_t = \sigma(b_o)\ c\sigma(b_1,\ldots,b_p)\kappa_t.$$

Since the original sequence is admissible, $b_1 < 2^m + 1$ and the result is immediate from the induction hypothesis.

Assume that $b_o = 2^{m+1} + 1$. Then

$$c\sigma(2^{m+1} + 1, b_1,\ldots,b_p)\kappa_t = \sigma(b_1 + 2^m)\ c\sigma(2^m + 1, b_2,\ldots,b_p)\kappa_t$$

$$+ \sum_{j=b_1+1}^{2^m} \gamma_j \sigma(2^{m+1} + b_1 + 1 - j)\ c\sigma(j, b_2,\ldots,b_p)\kappa_t.$$

Since $j > b_1$, the second summand consists of admissible sequences. The lemma follows immediately by induction.

Corollary 4.4. If $\sigma(a_1,\ldots,a_p)\kappa_t$ is an admissible generator of $L(U/O)$ and if $a_1 < 2^{p+1}r + 1$, then for each r such that $\frac{t}{2} \leq r \leq t - 1$, $c\sigma(a_1,\ldots,a_p, 2r + 1)\kappa_{t-r} = \zeta\sigma(a_1,\ldots,a_p)\kappa_t \bmod F_{t-1}$ where $\zeta = 1$ if $r = 2^i$ and $\zeta = 0$ otherwise.

Proof. By Lemma 4.2,

$$\sigma(a_1,\ldots,a_p, 2r + 1) = \sigma(2^{p+1}r + 1, a_1 - 2^p r,\ldots,a_p - 2r)$$

+ lower terms. Let i be the least integer such that $r \leq 2^i$. Thus $2^{p+1}r + 1 \leq 2^{p+i+1} + 1$ and the result is immediate from Lemma 4.3 with $m = p + i$.

Lemma 4.5. Assume that $\sum_i \sigma(a_i, b_i) = 0$ in $L(n)$. Then for all integers $m \geq u \geq 0$ we have $\sum_{s=u+1}^{m} \sum_i \sigma(a_i + 2^{m+1} - 2^s, b_i + 2^s - 2^u) = 0$.

Proof. In order to take advantage of some classical results, we prove the analogous theorem for the corresponding Steenrod relations. It can easily be checked that all computations remain in the image of $L(n)$. We would like to thank T. Bisson for suggestions regarding this proof.

First observe that for $b \geq 0$

$$Sq^{2^m+2b+1} Sq^{2^m-2^u+b+1} = \sum_k \binom{2^m-2^u+b-k}{2^m+2b+1-2k} Sq^{2^{m+1}-2^u+3b+2-k} Sq^k$$

$$= \sum_j \binom{2^m-j}{2^m+2^{u+1}+1-2j} Sq^{2^{m+1}-j+2b+2} Sq^{j-2^u+b}$$

$$= \sum_{s=u+1}^{m-1} Sq^{2^{m+1}-2^s+2b+1} Sq^{2^s-2^u+b+1}$$

by Lemma 4.1. Thus we have

$$A = \sum_{s=u+1}^{m} Sq^{2^m-2^s+2b+1} Sq^{2^s-2^u+b+1} = 0,$$

i.e. the lemma holds for the relation $\sigma(2b+1, b+1) = 0$.

The relations for $b \geq i \geq 0$, $B = \sum_t \binom{i}{t} Sq^{2b-i+t+1} Sq^{b-t+1} = 0$ are known to generate all relations in $\mathcal{a}$. (See [M3], [K2]). Thus it suffices to show that

$$C = \sum_{s=u+1}^{m} \sum_t \binom{i}{t} Sq^{2^{m+1}-2^s+2b-i+t+1} Sq^{2^s-2^u+b-t+1} = 0.$$

Let $\kappa:\mathcal{a} \longrightarrow \mathcal{a}$ be the derivation defined by $\kappa Sq^a = Sq^{a-1}$ [K2]. Thus if $\theta = 0$, so is $\kappa^i\theta$. For example, $B = \kappa^i Sq^{2b+1} Sq^{b+1}$ by the Leibnitz formula for the derivative of a product. The lemma follows by observing that $C = \kappa^i A$ by using the same Leibnitz formula.

Corollary 4.6. If $p > 1$ and $A = \sum_\alpha \sigma(b_1^\alpha, \ldots, b_p^\alpha) = 0$, then for any m and u with $m - u \geq p$ we have

$$\sum_{\langle S \rangle} \sum_\alpha \sigma(b_1^\alpha + 2^m - 2^{s_1}, \ldots, b_p^\alpha + 2^{s_{p-1}} - 2^u) = 0$$

where S ranges over sequences $(s_1, \ldots, s_{p-1})$ with $m > s_1 > \ldots > s_{p-1} > u$.

Proof. For $p = 2$ this is Lemma 4.5. Since all relations in L, as in $\mathcal{a}$, are generated by relations of length 2, we may write A as a sum of relations of the form

$$\sum_\alpha \sigma(b_1^{\alpha_o}, \ldots, b_{i-1}^{\alpha_o}, b_i^\alpha, b_{i+1}^\alpha, b_{i+2}^{\alpha_o}, \ldots, b_p^{\alpha_o}) = 0.$$

The result now follows from Lemma 4.5 by fixing $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$ and letting s_i vary subject to $s_{i-1} > s_i > s_{i+1}$.

Lemma 4.7. Assume that $\sigma(2^m + 1, a_2,\dots,a_p)\kappa_t$ is admissible in $L(2t - 1)$ and that $a_2 < 2^{p+i} + 1$ and $2^{i-1} < r \leq 2^i$. Then

$$c\sigma(2^m + 1, a_2,\dots,a_p, 2r + 1)\kappa_{t-r} =$$

$$\zeta\sigma(2^m + 1, a_2,\dots,a_p)\kappa_t + \sum_{m>s_1>\dots>s_{p-1}>u} \sum_{2^{u-1}>r} \sigma(a_2 + 2^m - 2^{s_1},\dots,2r + 1 + 2^{s_{p-1}} - 2^u)\kappa_{t-r+2^{u-1}}$$

modulo F_{t-1} where $\zeta = 1$ if $r = 2^i$ and $\zeta = 0$ if $r < 2^i$.

Proof. Since the original sequence is admissible and $a_p > 2r + 1$, we see that $m > p + i$. By Lemma 4.2 we may write

$$(4.8)\quad \sigma(a_2,\dots,a_p, 2r + 1) = \zeta\sigma(2^{p+i} + 1,\dots,a_p - 2^{i+1}) + \sum_\alpha \sigma(b_1^\alpha,\dots,b_p^\alpha)$$

where ζ is as above and $b_1^\alpha < 2^{p+i} + 1$ for each α. This is true since if $r < 2^i$, i.e. if $\zeta = 0$, then $2^{F_r} + 1 < 2^{p+i} + 1$ and thus appears in the second sum.

Consider the expression $A_\alpha = c\sigma(2^m + 1, b_1^\alpha,\dots,b_p^\alpha)\kappa_{t-r}$. Since $m > p + i$, this is admissible and so by Definition 3.6

$$A_\alpha = \sum_{2^{m-1}+1\geq j>b_1^\alpha} \gamma_j\sigma(2^m + 1 + b_1^\alpha - j)c\sigma(j, b_2^\alpha,\dots,b_p^\alpha)\kappa_{t-r}.$$

If j is not of the form $2^s + 1$, then $c\sigma(j, b_2^\alpha,\dots,b_p^\alpha)\kappa_{t-r} = \sigma(j)\ c\sigma(b_2^\alpha,\dots,b_p^\alpha)\kappa_{t-r} \in F_{t-1}$ by Lemma 4.3 since $b_2^\alpha < 2^{p+i-1} + 1$. If $j = 2^s + 1$, then $\gamma_j = 1$ by Lemma 4.1 and so

$$A_\alpha = \sum_{2^m+1>2^s+1>b_1^\alpha} \sigma(b_1^\alpha + 2^m - 2^s)c\sigma(2^s + 1, b_2^\alpha,\dots,b_p^\alpha)\kappa_{t-r}$$

mod F_{t-1}. By a simple induction argument on p,

$$A_\alpha = \sum_{m>s_1>\dots>s_{p-1}>u} \sigma(b_1{}^\alpha + 2^m - 2^{s_1},\dots,b_p{}^\alpha + 2^{s_{p-1}} - 2^u)c\sigma(2^u + 1)\kappa_{t-r} \mod F_{t-1}$$

$$= \sum_{\langle S\rangle,u} \sigma(b_1{}^\alpha + 2^m - 2^{s_1},\dots,b_p + 2^{s_{p-1}} - 2^u)\kappa_{t-r+2^{u-1}} \mod F_{t-1}.$$

Thus by Corollary 4.6 and equation (4.8)

$$\sum_\alpha A_\alpha = \sum_{\langle S\rangle,u} \sigma(a_2 + 2^m - 2^{s_1},\dots,2r + 1 + 2^{s_{p-1}} - 2^u)\kappa_{t-r+2^{u-1}}$$

$$+ \sum_{\langle S\rangle,u} \sigma(2^{i+p} + 1 + 2^m - 2^{s_1},\dots,a_p - 2^{i+1} + 2^{s_{p-1}} - 2^u)\kappa_{t-r+2^{u-1}}$$

mod F_{t-1}. If $\zeta = 0$, that is if $r < 2^i$, then the proof is complete. Otherwise, if $r = 2^i$, it remains to show that

$$(4.9)\quad \begin{cases} c\sigma(2^m + 1,\ 2^{p+i} + 1,\dots,a_p - 2^{i+1})\kappa_{t-2^i} \\ + \sum_{\langle S\rangle,u} \sigma(2^{i+p} + 1 + 2^m - 2^{s_1},\dots,a_p - 2^{i+1} + 2^{s_{p-1}} - 2^u)\kappa_{t-2^i+2^{u-1}} \\ = \sigma(2^m + 1,\ a_2,\dots,a_p)\kappa_t \mod F_{t-1}. \end{cases}$$

If $m = p + i + 1$, then the first term above vanishes. Moreover the only sequence $(s_1,\dots,s_{p-1},\ u)$ which satisfies the hypothesis for the second sum is $(i + p,\dots,i + 1)$. If we call $s_p = u$ we see that $s_{p-j} = i + j + 1$ and thus $-2^{i+j} + 2^{s_{p-j}} - 2^{s_{p-j-1}} = 0$, $j = 1,\dots,p - 1$, and so equation (4.9) is valid.

Finally assume that $m > i + p + 1$. It follows as above (see also Lemma 4.1) that

$$c\sigma(2^m + 1,\ 2^{i+p} + 1,\dots,a_p - 2^{i+1})\kappa_{t-2^i}$$

$$= \sum_{m>s>i+p} \sigma(2^{i+p} + 1 + 2^m - 2^s)c\sigma(2^s + 1,\dots,a_p - 2^{i+1})\kappa_{t-2^i} \mod F_{t-1}$$

$$(4.10)\quad \begin{cases} =\sigma(2^m + 1)c\sigma(2^{i+p}+1,\dots,a_p - 2^{i+1})\kappa_{t-2^i} \\ + \sum_{m>s\geq i+p} \sigma(2^{i+p} + 1 + 2^m - 2^s)c\sigma(2^s + 1,\dots,a_p - 2^{i+1})\kappa_{t-2^i} \end{cases}$$

mod F_{t-1}. The first term of (4.10) is equal to

$\sigma(2^m + 1, a_2,\ldots,a_p)\kappa_t$ mod F_{t-1}, i.e. the last expression in (4.9), by Lemma 4.3. The second expression of (4.10) equals the second expression of (4.9) by induction.

Corollary 4.11. Assume that $\sigma(2^m + 1, a_2,\ldots,a_p)\kappa_t \in L(U/O)$ is admissible and that $a_2 < 2^{i+p} + 1$ if $r \leq 2^i$. Then

$$c\sigma(2^m + 1, a_2,\ldots,a_p, 2r + 1)\kappa_{t-r} = \tag{4.12}$$
$$\sum_{2^{m-1}+1\geq j>a_2} \gamma_j\sigma(a_2 + 2^m - j + 1)\ c\sigma(j, a_3,\ldots,2r + 1)\kappa_{t-r}$$

mod F_{t-1}.

Note that this says that the definition of c extends to a certain class of nonadmissible sequences modulo F_{t-1}. Indeed the heart of the proof of Theorem 3.8 is that this fact is true in general.

Proof. First assume that $r < 2^i$. If $j \neq 2^s + 1$, then as in the proof of Lemma 4.7, and using Lemma 4.3, we see that $c\sigma(j, a_3,\ldots,2r + 1)\kappa_{t-r} = \sigma(j)c\sigma(a_3,\ldots,2r + 1)\kappa_{t-1} \in F_{t-1}$. If $j = 2^s + 1$, then $\gamma_j = 1$ by Lemma 4.1. Thus the right hand side of equation (4.12) equals

$$\sum_{2^m+1>2^s+1>a_2} \sigma(a_2 + 2^m - 2^s)c\sigma(2^s + 1, a_3,\ldots,2r + 1)\kappa_{t-r} \quad \text{mod } F_{t-1}.$$

The lemma now follows by expanding both sides using Lemma 4.7.

Now assume that $r = 2^i$. If $j \neq 2^s + 1$, then by Corollary 4.4, $c\sigma(j, a_3,\ldots,2^{i+1} + 1)\kappa_{t-2^i} = \sigma(j)c\sigma(a_3,\ldots,2^{i+1} + 1)\kappa_{t-2^i} = \sigma(j, a_3,\ldots,a_p)\kappa_t$. If $j = 2^s + 1$, then by Lemma 4.7 with $\zeta = 1$ we get $c\sigma(2^s + 1, a_3,\ldots,2^{i+1} + 1)\kappa_{t-2^i} = \sigma(2^s + 1, a_3,\ldots,a_p)\kappa_t + \sum_{\langle S\rangle,u} \sigma(a_3 + 2^s - 2^{s_2},\ldots,2^{i+1} + 1 + 2^{s_{p-1}} - 2^u)\kappa_{t-2^i+2^{u-1}}$. Thus the right hand side of (4.12) is equal to $\sum\gamma_j\sigma(a_2 + 2^m - j + 1, j, a_3,\ldots,a_p)\kappa_t + \sum_s \sum_{\langle S\rangle,u} \sigma(a_2 + 2^m - 2^s,\ldots,2^{i+1} + 1 + 2^{s_{p-1}} - 2^u)\kappa_{t-2^i+2^{u-1}}$ by Lemma 4.7.

Since $\sum_{j>a_2} \gamma_j \sigma(a_2 + 2^m - j + 1, j) = \sigma(2^m + 1, a_2)$, this is precisely the left hand side of (4.12) after expanding it by Lemma 4.7.

5. PROOF OF THEOREM 3.8. The proof of this theorem proceeds by induction on the length of the sequence I. If that length is 0, we must show that $(cd + dc)\kappa_t = \kappa_t \bmod F_{t-1}$. By definition, $c\kappa_t = 0$ and

$$(5.1) \qquad d\kappa_t = \sum_{r=[\frac{t}{2}]}^{t-1} \binom{r-1}{t-r-1} \sigma(2r + 1)\, \kappa_{t-r}.$$

Let i be the unique integer such that $\frac{t}{2} \leq 2^i < t$. Then it is easy to see that $\binom{2^i-1}{t-2^i-1} = 1 \pmod 2$ (see Lemma 2.6 [SE]), so that

$$d\kappa_t = \sigma(2^{i+1} + 1)\kappa_{t-2^i} + \sum_{\substack{r \neq 2^i \\ \frac{t}{2} \leq r < t}} \gamma_r \sigma(2r + 1)\kappa_{t-r}.$$ The result now follows

immediately from Definition 3.6

Now assume that the theorem is true if $\ell(I) < p$ and let $\alpha = \sigma(a, J)\kappa_t$ be admissible where $\ell(J) = p - 1$. To complete the proof we must consider two cases, each with several subcases.

Case 1. $a \neq 2^m + 1$

Clearly $dc\sigma(a, J)\kappa_t = d\sigma(a)c\sigma(J)\kappa_t =$

$$\sigma(a)\ dc\sigma(J)\kappa_t =$$

$$\sigma(a)[\sigma(J)\kappa_t + cd\sigma(J)]\kappa_t \bmod F_{t-1}$$

by induction. Thus we must show that $cd\sigma(a)\sigma(J)\kappa_t = \sigma(a)\ cd\sigma(J)\kappa_t \bmod F_{t-1}$. This certainly follows if we can show

$$(5.2) \quad c\sigma(a, J, 2r + 1)\kappa_{t-r} = \sigma(a)\ c\sigma(J, 2r + 1)\kappa_{t-r} \bmod F_{t-1}$$

for each r, $\frac{t}{2} \leq r < t$. Note that if $\sigma(a, J, 2r + 1)$ were admissible, this would follow from the definition of c. We prove (5.2) first for $a > 2^{p+1} r + 1$ and then for $a < 2^{p+i+1} + 1$ where $r \leq 2^i$. Since a is not of the form $2^m + 1$, these subcases are exhaustive.

Subcase 1.1. $a > 2^{p+1} r + 1$.

Write $\sigma(J, 2r + 1) = \sum\sigma(K_j)$ in admissible form and note that $c(a, J, 2r + 1) = \sum\sigma(a, K_j)$ in admissible form by Lemma 4.2. Then

$$\sum c\sigma(a, K_j)\kappa_{t-r} = \sum\sigma(a)\ c\sigma(K_j)\kappa_{t-r} \mod F_{t-1}$$

$$= \sigma(a)\ c\sigma(J, 2r + 1)\kappa_{t-r} \mod F_{t-1}$$

and we are done.

Subcase 1.2. $a < 2^{p+i+1} + 1$ where $r \leq 2^i$.

Since $\sigma(a, J)$ is admissible, Lemma 4.3 implies that each side is equal to $\zeta\sigma(a, J)\kappa_t \mod F_{t-1}$ where $\zeta = 1$ if $r = 2^i$ and $\zeta = 0$ if $r < 2^i$ and again we are done.

Case 2. $a = 2^m + 1$.

Write $J = (b, K)$. Then

$$dc\sigma(2^m + 1, b, K)\kappa_t =$$

$$\sum_{j=b+1}^{2^m+1} \gamma_j\sigma(2^m + b + 1 - j)\ dc\sigma(j, K)\kappa_t$$

$$= \sum_{j=b+1}^{2^m+1} \gamma_j\sigma(2^m + b + 1 - j)[\sigma(j, K)\kappa_t + cd\sigma(j, K)\kappa_t] \mod F_{t-1}.$$

Recall that $\sum_{j=b+1}^{2^m+1} \gamma_j\sigma(2^m + b + 1 - j, j) = \sigma(2^m + 1, b)$

is an Adem relation. Thus as in Case 1 it suffices to show that

$$cd\sigma(2^m + 1, b, K)\kappa_t =$$

$$\sum\gamma_j\sigma(2^m + 1 + b - j)\ cd\sigma(j, K)\kappa_t$$

or more simply that

$$(5.3)\quad c\sigma(2^m + 1, b, K, 2r + 1)\kappa_{t-r} = \sum\gamma_j\sigma(2^m + 1 + b - j)c\sigma(j, K, 2r + 1)\kappa_{t-r}$$

for each r, $\frac{t}{2} \leq r < t$. Again this follows immediately from Definition 3.6 if $\sigma(2^m + 1, b, K, 2r + 1)$ is admissible.

Subcase 2.1. $b > 2^p r + 1$.

Write $\sigma(2^m + 1, b, K, 2r + 1)$ in admissible form $\sum\sigma(2^m + 1, b, K_\alpha)$ and proceed exactly as in Case 1.1.

Subcase 2.2. $b < 2^{p+i} + 1$ where $r \leq 2^i$. This is precisely the case covered in Corollary 4.11.

If $r < 2^i$, then these subcases overlap and Case 2 is finished. Thus we need just one more special case.

Subcase 2.3. $r = 2^i$ and $b = 2^{i+p} + 1$.

By Lemma 4.1, $\gamma_j = 1$ in equation 5.3 if and only if $j = 2^s + 1$ and $2^{i+p} + 1 < j \leq 2^{m-1} + 1$. Thus we must show

$$(5.4) \quad \sigma(2^m + 1,\ 2^{i+p} + 1,\ K,\ 2^{i+1} + 1)\kappa_{t-2^i} = \sum_{s=i+p+1}^{m-1} \sigma(2^m + 2^{i+p} + 1 - 2^s)c\sigma(2^s+1,\ K,\ 2^{i+1} + 1)\kappa_{t-2^i}.$$

Let $\sigma(K) = \sigma(a_3,\ldots,a_p)$ and use Lemma 4.2 to write

$$(5.5) \quad \sigma(K,\ 2^{i+1} + 1) = \sigma(2^{i+p-1} + 1,\ a_3 - 2^{i+p-2},\ldots,a_p - 2^{i+1}) + \sum_\alpha \sigma(b_2{}^\alpha,\ldots,b_p{}^\alpha)$$

where $b_2{}^\alpha < 2^{i+p-1} + 1$ for all α.

Since $\sigma(2^{i+p} + 1,\ 2^{i+p-1} + 1) = 0$,

$$c\sigma(2^m + 1,\ 2^{i+p} + 1,\ K,\ 2^{i+1} + 1)\kappa_{t-2^i} =$$

$$\sum_\alpha c\sigma(2^m + 1,\ 2^{i+p} + 1,\ b_2{}^\alpha,\ldots,b_p{}^\alpha)\kappa_{t-2^i}$$

$$= \sum_{s=i+p+1}^{m-1} \sum_\alpha \sigma(2^m + 1 + 2^{i+p} - 2^s)c\sigma(2^s + 1,\ b_2{}^\alpha,\ldots,b_p{}^\alpha)\kappa_{t-2^i}$$

$$= \sum_{s=i+p+1}^{m-1} \sigma(2^m + 1 + 2^{i+p} - 2^s)[c\sigma(2^s + 1,\ a_3,\ldots,a_p,\ 2^{i+1} + 1)\kappa_{t-2^i} + c\sigma(2^s + 1,\ 2^{i+p-1} + 1,\ldots,a_p - 2^{i+1})\kappa_{t-2^i}].$$

Here we use Lemma 4.1 to evaluate γ_j and also formula (5.5) backwards. Thus to prove that (5.3) holds, it suffices to show that

$$\sum_{s=i+p+1}^{m-1} \sigma(2^m + 1 + 2^{i+p} - 2^s)c\sigma(2^s + 1,\ 2^{i+p-1} + 1,\ldots,a_p - 2^{i+1})\kappa_{t-2^i} = 0.$$

But by Lemma 4.1 again, this sum equals

$$\sum_{i+p-1<u<s<m} \sigma(2^m + 1 + 2^{i+p} - 2^s,\ 2^s + 1 + 2^{i+p-1} - 2^u)$$

$$c\sigma(2^u + 1,\ a_2 - 2^{i+p-2}, \ldots, a_p - 2^{i+1})\kappa_{t-2^i} = 0$$

by Lemma 4.5 applied to the relation $\sigma(2^{i+p} + 1,\ 2^{i+p-1} + 1) = 0$.

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